Grade 6 Math Circles
March 5 & 6 & 7, 2024
Math Paradoxes

Paradoxes

Think and Share. In your own words, what's a paradox?

The Paradox of Standing and Walking

For many people, it is more tiring to stand in one spot then it is to walk. How can this be? Walking involves more movement so it makes more sense for walking to require a greater amount of energy. This is *paradoxical*!

A paradox is a statement, fact, or observation that appears counter-intuitive (i.e., surprising). When presented with a paradox, many of us say, "That can't be right!" and ask, "How can that be?"

To **resolve a paradox** means to come up with an explanation or way of understanding a paradox that helps it make more sense to us. Paradoxes often highlight genuine problems in our understanding of certain (philosophical) concepts. By trying to resolve a paradox, we can learn more about the world and our understanding of it.

Word Paradoxes

Sometimes a word can mean both one thing and its opposite. Such a word is called a contronym or an auto-antonym.

For example, the word *to dust* can mean to remove dust or to sprinkle something with dust, such as in "I was dusting the vase" (i.e., removing dust from it) and "I was dusting the surface with flour" (i.e., putting flour on).

As another example, the word "aloha" can mean both hello and goodbye. To resolve this paradox, we can consider that the Hawaiian origin of the word "aloha" means love, and so it is used as an expression of good wishes upon either greeting or parting.

Word paradoxes such as these can often be resolved by considering the context around which the word is used.

Math Paradoxes

Now that we know what paradoxes are, let's take a look at paradoxes that involve mathematics.

Coin Rotation Paradox

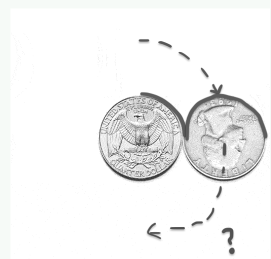
Suppose that there are two coins of the same radius, r, laid on a table. The coins are touching each other at exactly one point. If we were to rotate the outer coin around the center coin all the way around back to where it started, how many rotations would the outer coin undergo?

How many times do you think the outer coin would rotate as it goes fully around the circumference of the inner coin?

My prediction is: ______________________________.

Let's rotate the outer coin around the inner one and see what happens. When we roll the outer coin one-fourth of the way around the inner coin, it has already rotated 180°! It is completely upside down. Strangely, half of the circumference of the outer coin has been traversed, even though only a quarter of the circumference of the inner coin has been traversed.

What do you think would happen if we continue to rotate the outer coin so that it is directly below the inner coin? How many turns do you think the outer coin would have made?

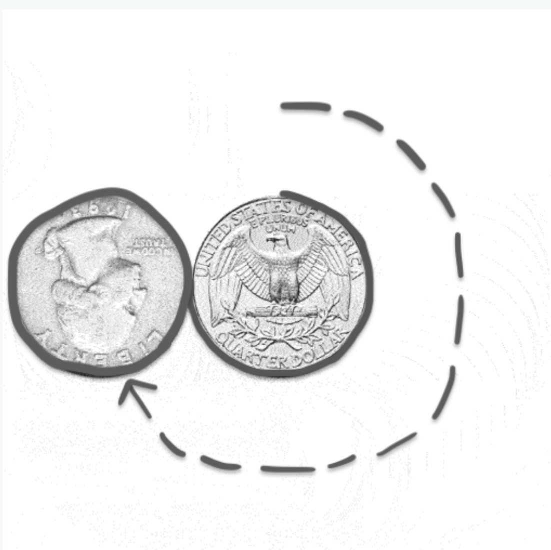


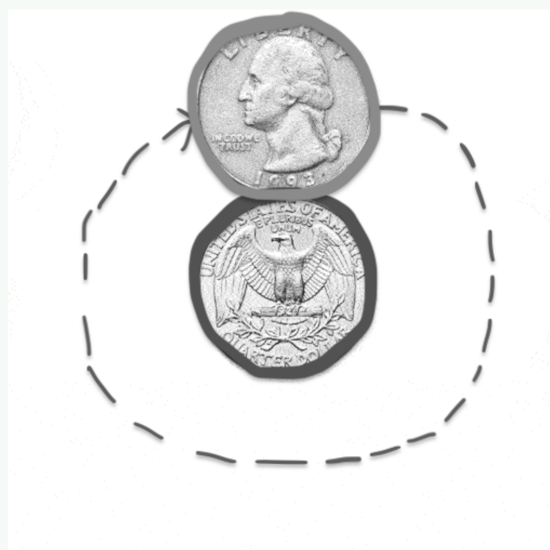
My prediction is: ________

When we continue to rotate the outer coin so that it is directly below the inner coin, we find that the outer coin is now the right side up again, meaning that the outer coin has made a complete turn! Now, the entire circumference of the outer coin has been traversed, even though only half of the circumference of the inner coin has been traversed.

As we rotate the outer coin three-fourths of the way around the circumference of the inner coin, the outer coin has now completed $1\frac{1}{2}$ turns.

Finally, when the outer coin is rotated all the way around the circumference of the inner coin, the outer coin has made two full turns!





Key fact: The outer coin makes two rotations rolling once around the inner coin.

This seems paradoxical because only the circumference of the inner coin has been traversed once whereas two times the circumference of the outer coin has been traversed. Recall that the coins are the same size and thus have the same circumference.

To resolve this paradox, let's consider the path that the center of the outer coin travels. Recall that r represents the radius of either of the two coins (which have the same radii). What is the radius of the circular path on which the center of the outer coin travels as it rolls once around the inner coin (*pick one of the options below*)?

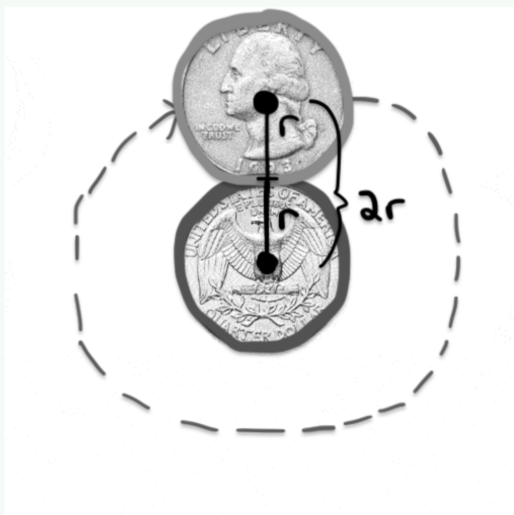
a) r

b) $\frac{1}{2}r$

c) $1\frac{1}{2}r$

d) $2r$

The correct answer is $2r$. We can see this visually in the diagram below.



Throughout analysis of the problem, we will use the fact that *radius and circumference are proportional to each other*. Later in your mathematical education, you will see that the circumference of a circle is always 2π times its radius. For example,

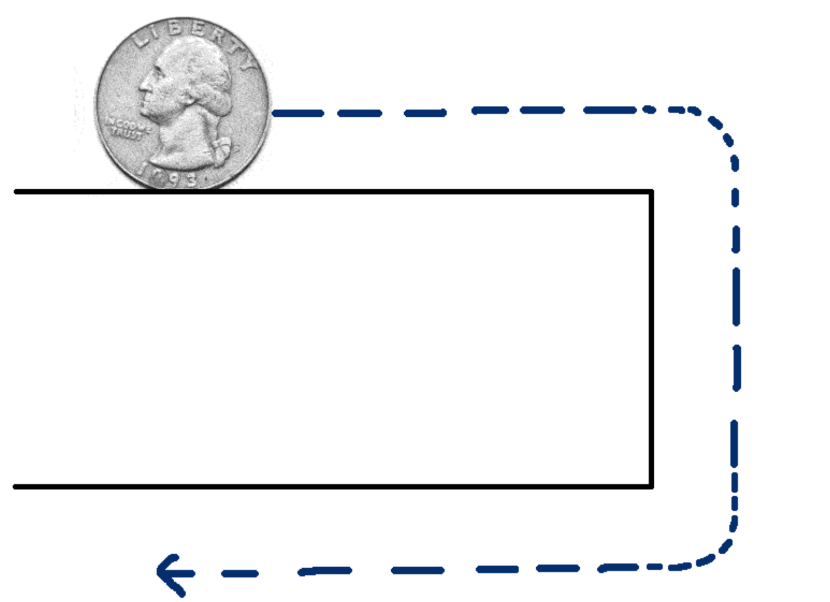
- If the radius of one coin is the same as another coin, then the two coins have the same circumference.
- If the radius of one coin is twice the radius of another coin, then the circumference of the first coin is twice the circumference of the second.
- etc...

Stop and Think. If the radius of Circle A is 5 times the radius of Circle B, how many times longer is the circumference of Circle A compared to the circumference of Circle B?[a]

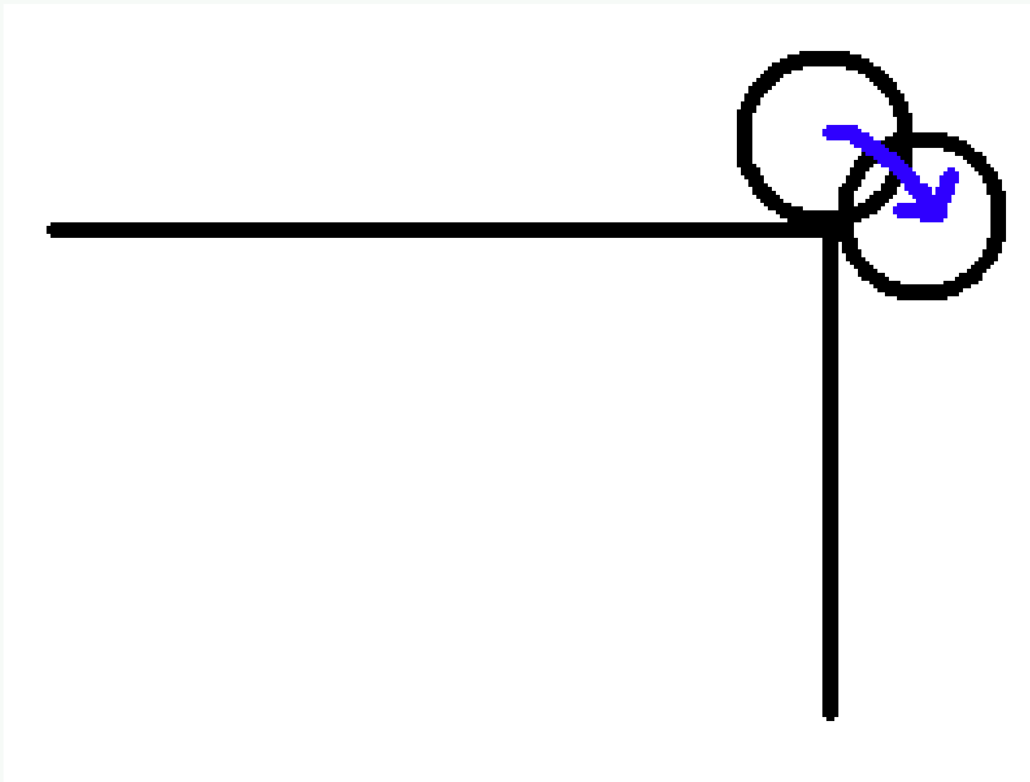
If the circumference of Circle A is 2 times the circumference of Circle B, how many times longer is the radius of Circle A compared to Circle B?[b]

So the center of the outer coin travels on a circular path whose circumference is two times the circumference of the outer coin. In effect, the outer coin is rolling a distance that is two times its circumference, so it makes two full turns!

Of course, the coin is not actually rolling on a path whose distance is two times its circumference. The coin is rolling along the circumference of the inner coin, which has the same length as the circumference of the outer coin. The extra rotation comes from the fact that the outer coin is rolling on a circle! When viewed from an external frame of reference, the circular shape of this path adds one additional rotation to the outer coin. Imagine that the outer coin rolled instead on a path that looks like a cliff that then goes upside down.



This distance that the center of the coin travels (also called the *path length*) may be longer when the rolling surface is not straight. For example, in our cliff path example, the coin makes an extra quarter turn every time that it hits a corner, even as its contact point on the surface is stationary throughout the corner turn. In total, there would be a half of a turn added to the number of turns that the coin *would* make if the path were straight. This is because the coin goes from sitting on top of the surface to hanging below it.



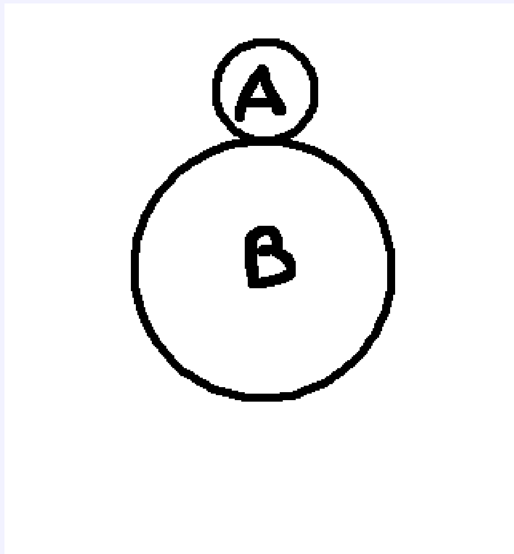
This discrepancy would be eliminated by taking the length of the path that the *center of the coin* travels and dividing this distance by the circumference of the coin. This method works every time.

[a] *Answer:* The circumference of Circle A is 5 times the circumference of Circle B.
[b] *Answer:* The radius of Circle A is 2 times the radius of Circle B.

Now that you've learned about the coin rotation paradox, let's test your understanding and see if you can figure out how to apply the principles of that we have described above.

Coin Rotation Paradox Exercise

In the figure below, the radius of Circle B is 3 times the radius of Circle A. Circle A rolls around Circle B one trip back to its starting point. How many times will Circle A revolve in total?



Solution 1: Since the radius of Circle B is 3 times the radius of Circle A, then the circumference of Circle B must be 3 times the circumference of Circle A as well.

In other words, Circle A rolls a distance that is three times its circumference. If the rolling path were straight, then Circle A would make 3 turns. Since the rolling path is a circle, then Circle A would make one additional turn. Therefore, Circle A makes $3 + 1 = 4$ turns as it rolls around Circle B.

Solution 2: Let r_a be the radius of Circle A and r_b be the radius of Circle B. The circular path that the center of Circle A travels on has a radius of $r_a + r_b$, which is $\frac{r_a+r_b}{r_a}$ times its own radius. Therefore, the circumference of this circular path is $\frac{r_a+r_b}{r_a}$ times the circumference of Circle A. From the question statement, we know that $r_b = 3r_a$. Substituting, we get

$$\begin{aligned}\frac{r_a + r_b}{r_a} &= \frac{r_a + 3r_a}{r_a}\\ &= \frac{4\cancel{r_a}}{\cancel{r_a}}\\ &= 4.\end{aligned}$$

Stop and Think. Which method do you like more? Do you prefer to calculate the ratio of the circumference lengths and then add 1 (*Solution 1*)? Or do you prefer to reason with the length of the path that the center of Circle A travels on (*Solution 2*)?

In summary, for a coin of radius r rolling around another coin of radius R, the coin with radius r makes $\boxed{\frac{R}{r}+1 \text{ or } \frac{R+r}{r} \text{ rotations.}}$

Statistical Paradoxes

The next paradox we will investigate is a **statistical paradox**. A statistical paradox reveals how the presentation, interpretation, or analysis of data can sometimes be misleading.

You've already learned some statistics in school. For example, the *mean* of a set of numbers can tell us about the central or "typical" value for that group of numbers.

An example of a counter-intuitive result involving the mean is the **Will Rogers Phenomenon**.

Let's say that you have the following two groups of numbers and that we move the lowest number from one of the groups to the other. Calculate the mean of both groups before and after the move[1].

Group 1	$1, 2, 3, 4$	Mean =	$\longrightarrow$	Group 1	$1, 2, 3, 4, 5$	Mean =
Group 2	$5, 6, 7, 8, 9$	Mean =		Group 2	$6, 7, 8, 9$	Mean =

What do you notice about the means of the groups before and after moving the number 5 from Group 2 to Group 1?

Sample Answer: After moving the number 5 from Group 2 to Group 1, the means of both groups increase.

Since the means of both groups have increased, it seems as if numerical quantity has been created. For example, if the numbers represent the intelligence score of two groups of people, it appears as if intelligence has been created. This is known as the Will Rogers Phenomenon.

To resolve this paradox, notice that the average of all the numbers across the two groups is the same, and the weighted average of the means of both groups is the same number as well.

Group 1	$1, 2, 3, 4$	Mean $= 2.5$	$\longrightarrow$	Group 1	$1, 2, 3, 4, 5$	Mean $= 3$
Group 2	$5, 6, 7, 8, 9$	Mean $= 7$		Group 2	$6, 7, 8, 9$	Mean $= 7.5$
Combined	$1, 2, 3, 4, 5, 6, 7, 8, 9$	Mean $= 5$		Combined	$1, 2, 3, 4, 5, 6, 7, 8, 9$	Mean $= 5$

When we move the numbers between the groups, the sizes of the groups change. The group size

[1] *Answer:* The mean for group 1 is 2.5 before the change and 3 after the change. The mean for group 2 is 7 before the move and 7.5 after the move.

affects the weighted average of the means of the groups, which is always equal to the mean of the combined group.

Here's a calculation of the weighted average of the two groups before the move:

$$\text{Mean} = \frac{2.5 \times 4 + 7 \times 5}{9} = 5.$$

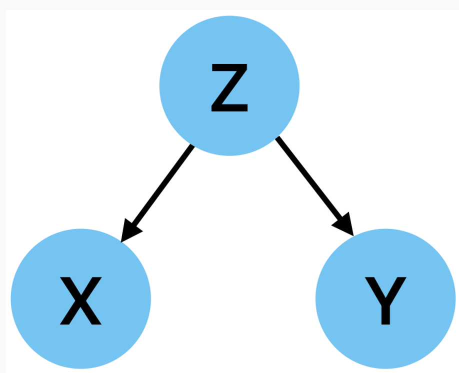
Here's a calculation of the weighted average of the two groups after the move:

$$\text{Mean} = \frac{3 \times 5 + 7.5 \times 4}{9} = 5.$$

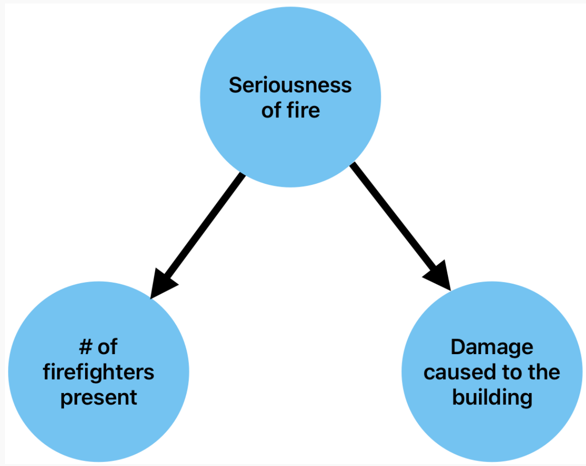
Both of these weighted averages is equal to the mean of the combined group containing all of the numbers.

Lurking Variables

In statistics, a lurking variable (also called a confounding variable) is something that acts as the common cause of two other variables.



For example, the seriousness of a fire can increase both the number of firefighters present and the damage caused to a building. Without considering this lurking variable, it may appear as if more firefighters present is one of or the only cause of damage to the building.



Stop and Think. A greater number of ice cream sales seems to cause an increase in shark attacks. What might a lurking variable be?

Sample Answer: A lurking variable might be the temperature. As temperature increases, more ice cream is sold and more shark attacks occur.

Stop and Think. Hot chocolate and sweater sales increase at the same time. Would you conclude that hot chocolate sales *cause* sweater sales? What might a lurking variable be.

Sample Answer: No, it does not make sense to conclude that hot chocolate sales cause sweater sales as there is no reasonable mechanism for that to happen. More probably, a lurking variable such as the colder temperature is responsible for this observation.

Simpson's Paradox

Lurking variables can lead to seemingly counterintuitive results.

For example, a study into two treatments for kidney stones involved 700 patients, with the following results:

- Treatment A: 273/350 successful treatments (78%)
- Treatment B: 289/350 successful treatments (83%)

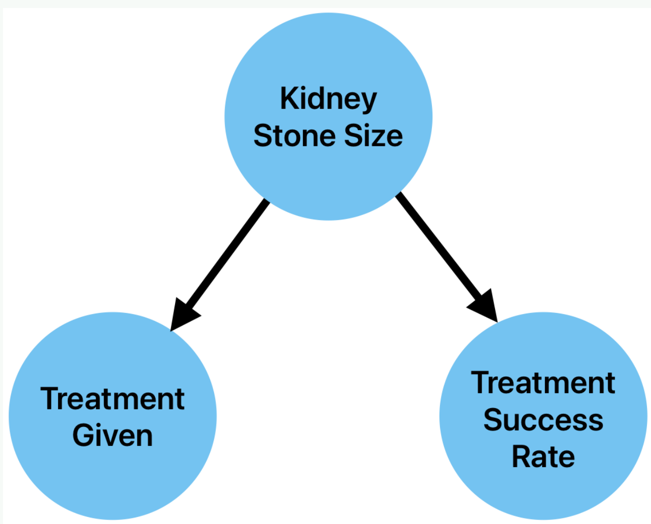
It appears as if Treatment B is better. However, the treatments weren't assigned at random! If the kidney stones are categorized as 'small' or 'large,' we see the following.

	Treatment A	Treatment B
Small stones	**93% (81/87)**	87% (234/270)
Large stones	**73% (192/263)**	69% (55/80)
Both	78% (273/350)	**83% (289/350)**

Treatment A is better in both groups!

Stop and Think. How can Treatment A be more effective for both small stones and large stones but Treatment B appear more effective when the data is aggregated?

This is because the size of the kidney stone is a lurking variable. When the data is aggregated, the lurking variable is hidden away.



The size of kidney stones has a big effect on the severity of the disease — and thus the success rate of treatment. The size of the stones also has an effect on which treatment doctors are likely to give the patient. Smaller stones are more often treated with the inferior Treatment B and larger stones are more often treated with the better Treatment A.

Since harder-to-treat cases are given the better treatment more often, whereas easier-to-treat cases are more often given the worse treatment, it appears that the worse treatment is better overall — but that's only because the worse treatment was used on easier cases more often.

This is an example of **Simpson's paradox**, which is a situation in statistics when one trend appears in groups of data but disappears or reverses when the groups are combined. In the kidney stone example, what are the groups of data? What was the trend that we observed within these groups? What trend did we see when the groups are combined?[a]

Lurking variables show up in many places and can be difficult to spot. Two common causes of Simpson's Paradox are age and the severity of the disease.

[a] *Answer:* The groups of data are patients with small kidney stones and patients with large kidney stones. Within each group, Treatment A worked better. When the groups were combined, Treatment B appeared more effective.

Simpson's Paradox Exercise

Imagine that a population of people in the United States consists of people who identify themselves as either "White/Asian American" or "Non-White/Non-Asian American." In our hypothetical population, 500 identify as "White/Asian American" and another 500 identify as "Non-White/Non-Asian American."

Now suppose that there is a pandemic. To make our numbers easier to work with, imagine that the pandemic has a high death toll (it kills a lot of people).

Here is a table showing the number of people and the number of deaths in each age group for people who identify as "White/Asian American."

- Complete the Mortality Rate column by dividing the deaths by the number of people within each age group. Then, add up all the numbers in the "Deaths" column to find the total number of deaths in the entire population of 500 people. Divide the total deaths by the total number of people in the population to find the overall mortality rate. **Express your answers for mortality rates as a percentage rounded to one decimal place**[a].

Age	Deaths	Mortality Rate	Total # of People in Population
0-9	1		20
10-19	5		55
20-29	15		100
30-39	55		205
40-49	35		60
50-59	18		30
60-69	7		10
70-79	4		5
80-89	8		10
90+	4		5
Overall			500

Here is a table showing the number of people and the number of deaths in each age group for people who identify as "Non-White/Non-Asian American".

- Complete the Mortality Rate column by dividing the deaths by the number of people within each age group. Then, add up all the numbers in the "Deaths" column to find the total number of deaths in the entire population of 500 people. Divide the total deaths by the total number of people in the population to find the overall mortality rate. **Express your answers for mortality rates as a percentage rounded to one decimal place**[b].

Age	Deaths	Mortality Rate	Total # of People in Population
0-9	0		10
10-19	1		25
20-29	5		40
30-39	10		50
40-49	28		60
50-59	44		85
60-69	50		85
70-79	50		75
80-89	35		55
90+	6		10
Overall			500

Overall, there are (*circle one*) more/less[c] deaths among the 500 people who identify as "White/Asian American" than the 500 people who identify as "Non-White/Non-Asian". However, when grouped by age, a (*circle one*) larger/smaller[d] proportion of people who identify as "White/Asian American" die from the disease compared to people who identify as "Non-White/Non-Asian American" within each age group.

Based on these results, would the headline "The death rate for White/Asian Americans now exceeds that of Non-White/Non-Asian Americans, proving the success of well-funded public health campaigns that narrow racial gaps" be misleading? Why or why not?[e]

This is an example of Simpson's paradox because the trend that appears in the overall population is different from the trend that appears when the population is grouped by age.

[a] *Answer:* The mortality rate for each age group is as follows: 5% (ages 0-9), 9.1% (ages 10-19), 15% (ages 20-29), 26.8% (ages 30-39), 58.3% (ages 40-49), 60% (ages 50-59), 70% (ages 60-69), 80% (ages 70-79), 80% (ages 80-89), 80% (ages 90+). The total number of deaths is 152. Therefore, the overall mortality rate is $\frac{152}{500} = 30.4\%$

[b] *Answer:* The mortality rate for each age group is as follows: 0% (ages 0-9), 4% (ages 10-19), 12.5% (ages 20-29), 20% (ages 30-39), 46.7% (ages 40-49), 51.8% (ages 50-59), 58.8% (ages 60-69), 66.7% (ages 70-79), 63.6% (ages 80-89), 60% (ages 90+). The total number of deaths is 229. Therefore, the overall mortality rate is $\frac{229}{500} = 45.8\%$

[c] *Answer:* More.

[d] *Answer:* Smaller.

[e] *Sample answer:* Yes, it can be misleading because this fails to account for the fact that White/Asian Americans are older in our population and probably have more severe symptoms that lead to higher death rates. Even though more White/Asian Americans are dying, it would be difficult to justify that this is due to more equitable vaccination efforts, etc., given that a smaller proportion of White/Asian Americans are dying within every age group. (When the lurking variable age is taken into account, the trend in mortality rate reverses.)

The Birthday Problem

Suppose that there are $n = 3$ people in a room: Alice, Bob, and Charles. What is the probability that there is a pair of people in this room who share the same birthday? How many pairs of people does one have to check in order to see if two of the three people share a birthday[2]?

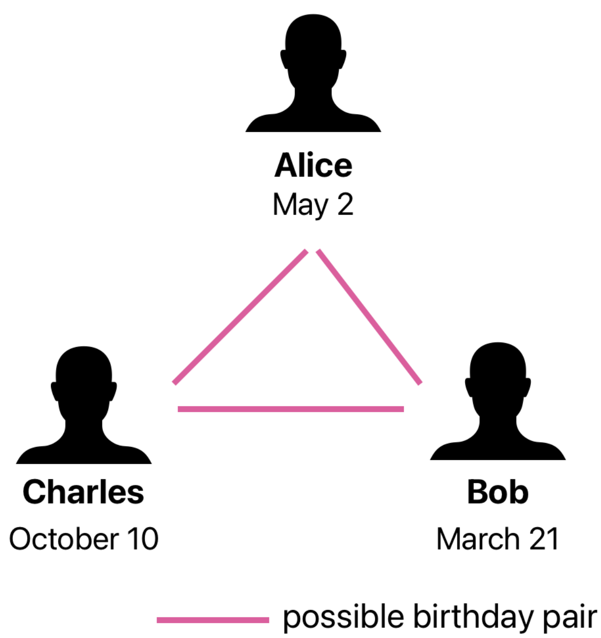


Figure 1: Illustration of the Birthday Problem with $n = 3$ People

Suppose now that there are $n = 4$ people in a room.. How many pairs of people would we have to check in order to see if two of the people in the room share a birthday[3]? What is the probability that there is a pair of people in this room who share the same birthday?

[2] We have to check 3 pairs of people: between Alice and Bob, between Bob and Charles, and between Alice and Charles

[3] There are 6 pairs of people we have to check (see Figure 2).

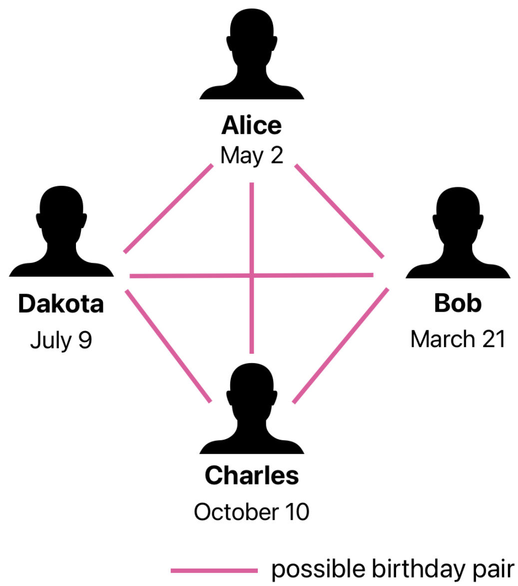


Figure 2: Illustration of the Birthday Problem with $n = 4$ People

Suppose now that there are $n = 5$ people in a room. How many pairs of people would we have to check in order to see if two of the people in the room share the same birthday[4]? What is the probability that there is a pair of people in this room who share the same birthday?

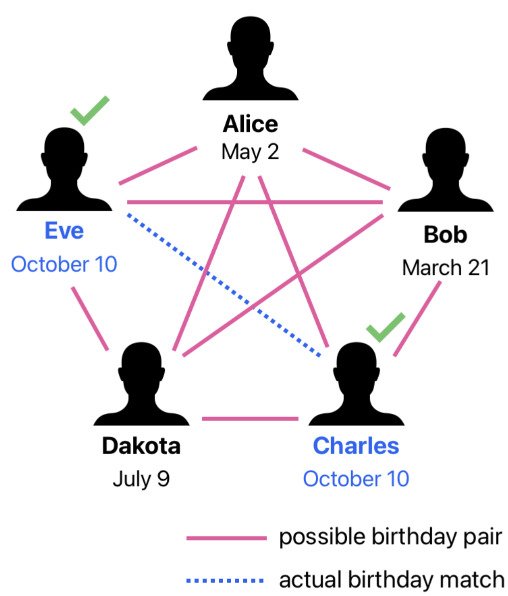


Figure 3: Illustration of the Birthday Problem with $n = 5$ People. The blue dotted line shows a pair of people with matching birthdays.

[4] There are 10 pairs of people we have to check (see Figure 3)

The **birthday problem** refers to finding the probability that, in a room with n people, at least two people in the room have the same birthday. We will calculate this probability shortly for every positive integer value of n.

Stop and Think. Assuming that each birthday is equally likely and no one was born in a leap year, what do you think is the probability that a room of $n = 23$ people contains two people who have the same birthday?

The Birthday Problem

Imagine that there are n people in a room. What is the probability that at least two people in the room share the same birthday? Assume that there are only 365 birthdays possible (ignore leap years) and that each person's birthday is equally likely to be any of these birthdays.

To calculate this probability, we first calculate the probability of finding n people without any repeated birthdays, and then subtract this from 100%.

$$100\% - \frac{\text{\# of ways to assign birthdays to each of the } n \text{ people with no repeats}}{\text{\# of ways to assign birthdays to each of the } n \text{ people}}$$

Since the first person can have any one of 365 birthdays, the second person can have any one of 365 birthdays, and so on, all the way until we reach the last, nth, person, the total number of outcomes is

$$\underbrace{365 \times 365 \times \cdots \times 365}_{n \text{ times}} = 365^n.$$

Also, if there are no repeated birthdays, then the first person can have any one of 365 birthdays, the second person can have any one of 364 birthdays, and so on, all the way until we reach the last, nth, person, who can have any one of the $(365 - n)$ birthdays not already present. So the total number of groups of n people (order matters) who do not share a birthday is

$$\underbrace{365 \times 364 \times \cdots \times (365 - n)}_{n \text{ people}} = \frac{365!}{(365 - n)!},$$

where the right hand side of the equation above is the same as the left hand side expressed using *factorials* so that you can put it into your scientific calculator.

Therefore, the probability that two people share a birthday in a room of n people is

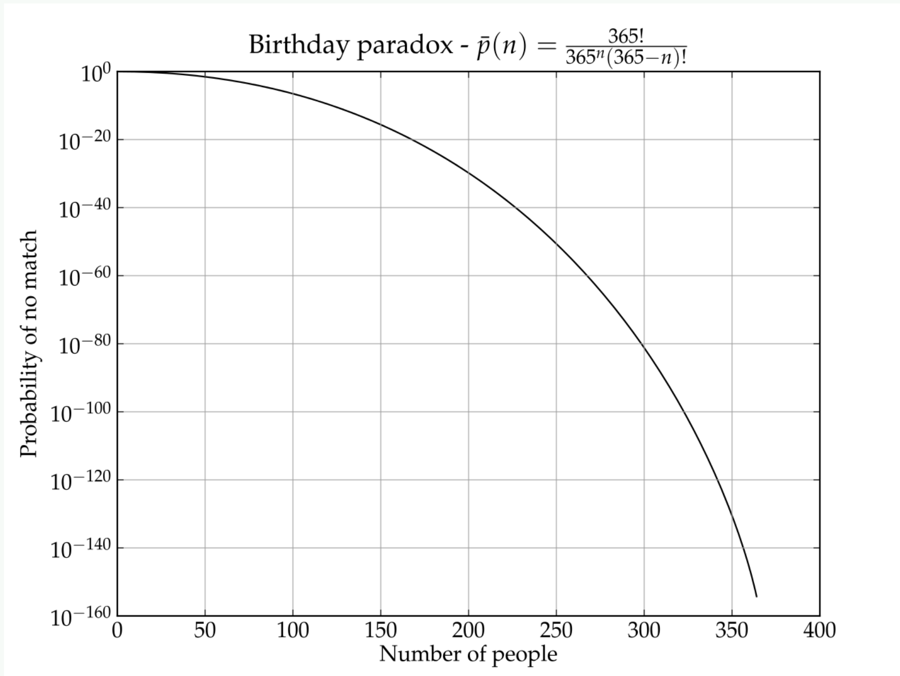
$$\begin{aligned}
&100\% - \text{probability that two people share a birthday} \\
&= 100\% - \frac{\#\text{ of ways to assign birthdays to each of the } n \text{ people with no repeats}}{\#\text{ of ways to assign birthdays to each of the } n \text{ people}} \\
&= 100\% - \frac{\frac{365!}{(365-n)!}}{365^n} \\
&= 100\% - \frac{365!}{(365-n)!365^n}
\end{aligned}$$

Using an online calculator such as WolframAlpha, we obtain

1. For $n = 10$ people, the probability is approximately 11.7%.
2. For $n = 23$ people, the probability is approximately 50.7%.
3. For $n = 35$ people (this is about how many people are in an in-person Math Circles session), the probability is approximately 70.6%.
4. For $n = 70$ people, the probability is approximately 99.9%.

So with just 23 people in a room, the probability that two people have the same birthday exceeds 50%! With just 70 people, this probability exceeds 99.9%. Many people would guess that these probabilities are much lower.

Here is a graph that shows the probability of there *not* two people who sharing the same birthday, which decreases steeply as n increases[a]. Note that each step down is 10^{20} times less likely.



[a] **Source:** Guillaume Jacquenot, CC BY-SA 3.0, via Wikimedia Commons